MUC-NOTE-COOL_THEORY-291
Muon Tech. Note MU-054

Examination of the Emittance Growth in Drifts

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(Dated: March 31, 2004)

Abstract

In a previous paper [1] I showed that ϵ_T and ϵ_6 growth in a drift as calculated by **ECALC9** [2]; subsequently, J.S Berg [3] has argued that this emittance growth is caused because a drift is not a linear element. We show that the non-linearity of a drift is an *apparent effect* due to the particular choice of the **Hamiltonian** function.

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I. INTRODUCTION

The standard **Hamiltonian**, for a drift, in accelerator physics is

$$-p_s = -\sqrt{(\frac{E}{c})^2 - (mc)^2 - p_x^2 - p_y^2} \equiv -\mathbf{H}(x, p_x, y, p_y, t, E; s); \tag{1}$$

we show explicitly the canonical variables and the independent variable s. The equation of motion are

$$\begin{aligned} \frac{dx}{ds} &= \frac{\partial \mathbf{H}}{\partial p_x} \equiv \frac{p_x}{\sqrt{(\frac{E}{c})^2 - (mc)^2 - p_x^2 - p_y^2}}, \\ \frac{dy}{ds} &= \frac{\partial \mathbf{H}}{\partial p_y} \equiv \frac{p_y}{\sqrt{(\frac{E}{c})^2 - (mc)^2 - p_x^2 - p_y^2}}, \\ \frac{dt}{ds} &= \frac{\partial \mathbf{H}}{\partial E} \equiv \frac{E}{c^2\sqrt{(\frac{E}{c})^2 - (mc)^2 - p_x^2 - p_y^2}}, \end{aligned} \tag{2}$$

which can be integrated exactly (canonical transformation). Obtaining

$$\begin{aligned} x(s) &= x(0) + \frac{p_x s}{\sqrt{(\frac{E}{c})^2 - (mc)^2 - p_x^2 - p_y^2}}, \\ y(s) &= y(0) + \frac{p_y s}{\sqrt{(\frac{E}{c})^2 - (mc)^2 - p_x^2 - p_y^2}}, \\ t(s) &= t(0) + \frac{E s}{c^2\sqrt{(\frac{E}{c})^2 - (mc)^2 - p_x^2 - p_y^2}}. \end{aligned} \tag{3}$$

J.S. Berg [3] very convincingly showed that, starting with this **Hamiltonian**, the equation of motion are non-linear (the square root in Eqs. 3) in the canonical variables and consequently, we can not expect the emittance to be constant.

II. A RELATIVISTIC Lagrangian FUNCTION OF 4-VELOCITY

It is known that the **Lagrangian** function and of course the **Hamiltonian** are not unique, the necessary requirements are: a) The Euler-Lagrange equations must be either the Newton equation for free space, or the Lorentz equation for a charged particle in an external field; and b) both functions must be Lorentz invariant.

An alternative approach that gives a quadratic **Lagrangian** and **Hamiltonian**, is briefly described in a problem by Jackson [4] (se also [5]). The **Lagrangian** is

$$\mathbf{L} = \frac{1}{2} m_o U_\mu U_\mu + \frac{q}{c} A_\mu U_\mu, \tag{4}$$

where $x_\mu = (\vec{x}, ict)$ is the 4-position, $U_\mu = (\gamma \vec{v}, i\gamma c) \equiv \frac{dx_\mu}{d\tau}$ is the 4-velocity, $d\tau = \frac{dt}{\gamma}$ is the proper time and $A_\mu = (\vec{A}, i\Phi)$ is the covariant vector potential.

The definition of 4-momentum is

$$p_\mu \stackrel{\text{def.}}{=} \frac{\partial \mathbf{L}}{\partial U_\mu} = m_o U_\mu + \frac{q}{c} A_\mu \tag{5}$$

with $p_\mu = (\vec{p}, ip_0)$. The **Hamiltonian** function is

$$\mathbf{H}(\vec{x}, \vec{p}, ct, p_0; \tau) \stackrel{\text{def.}}{=} p_\mu U_\mu - \mathbf{L} = \frac{1}{m_o} (p_\mu - \frac{q}{c} A_\mu)(p_\mu - \frac{q}{c} A_\mu); \tag{6}$$

we show explicitly the canonical variables and the independent variable τ. The equation of motion, for a drift, are:

$$\begin{aligned} \frac{d\vec{x}}{d\tau} = \frac{\vec{p}}{m_o}, \qquad & \frac{dct}{d\tau} = \frac{p_0}{m_o}, \\ \frac{d\vec{p}}{d\tau} = 0, \qquad & \frac{dp_0}{d\tau} = 0. \end{aligned} \tag{7}$$

Clearly, these equations can be integrated exactly and we get,

$$\begin{aligned} x(\tau) = x(0) + \frac{p_x}{m_o}\tau \quad &, \quad y(\tau) = y(0) + \frac{p_y}{m_o}\tau \\ z(\tau) = z(0) + \frac{p_z}{m_o}\tau \quad &, \quad ct(\tau) = ct(0) + \frac{E}{m_o c}\tau \\ \vec{p} = \text{constant} \quad &, \quad p_0 = \text{constant}. \end{aligned} \tag{8}$$

To compare with Eqs. 3 we note that $s = z(\tau) - z(0) = \frac{p_z}{m_o}\tau$ which implies $\tau = \frac{m_o s}{p_z}$; substituting in Eqs. 8 we can write

$$\begin{aligned} x(s) \equiv x(\tau) = x(0) \quad &+ \quad \frac{p_x}{m_o}\frac{m_o}{p_z} s, \\ y(s) \equiv y(\tau) = y(0) \quad &+ \quad \frac{p_y}{m_o}\frac{m_o}{p_z} s, \\ ct(s) \equiv ct(\tau) = ct(0) \quad &+ \quad \frac{E}{m_o c}\frac{m_o}{p_z} s, \\ \vec{p} = \text{constant}, \quad & \quad p_0 = \text{constant}. \end{aligned} \tag{9}$$

Formally Eqs. 9 are identical to Eqs. 3, however, because our phase space is 8-D rather than 6-D, the equation of motion for a drift are **linear** in the canonical variables $(\vec{x}, ct, \vec{p}, p_0)$

with independent variable τ. The emittances will be constant of motion when evaluated at constant τ planes. If the question we ask is: what is the emittance at constant s planes?, the answer will be: it is not a constant of motion unless there is no energy spread, because in such a case there is a one to one correspondance between τ and s for all particles.

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[1] J.C. Gallardo and R. Fernow,*Behavior of Kinematic Invariants of Beams with Large Energy Spread* MUC-NOTE-COOL_THEORY-288.

[2] R. Fernow, *ICOOL: A simulation code for Ionization Cooling of Muon Beam*, Proc. Particle Accelerator Conference, 1999. http://pubweb.bnl.gov/people/fernow/icool/readme.html.

[3] J.S. Berg, *Understanding Emittance Growth in Drifts*, Ring/Cooler/Emittance Exchange Workshop (2004). http://www.cap.bnl.gov/mumu/conf/E-EX-040127/.

[4] J.D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, New York 1962.

[5] Y.K. Wu, E. Forest and D.S. Robin, *Explicit symplectic integrator for s-dependent static magnetic field*, Phys. Rev. E**68**, 046502 (2003).